MATH4210: Financial Mathematics Tutorial 6

Xiangying Pang

The Chinese University of Hong Kong

xypang@math.cuhk.edu.hk

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Content Review

Question

(Distribution of Brownian motion) $\Rightarrow \frac{B_T}{\sqrt{T}} \sim N(0,1)$

$B_T - \underbrace{B_0}_{=0} \sim N(0,T)$

$$B_t - B_s \sim N(0, t-s)$$

Ex: $P(e^{B_T} > k)$
$= P(B_T > \ln k)$
$= P(\frac{B_T}{\sqrt{T}} > \ln k/\sqrt{T})$ ← $N(0,1)$
$= 1 - P(\frac{B_T}{\sqrt{T}} \le \ln k/\sqrt{T})$
$= 1 - \Phi(\ln k/\sqrt{T})$

$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

$dS_t = \mu S_t dt + \sigma S_t dB_t$

Question

S_t follows the Black Scholes Model with drift μ and volatility σ, we have

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t).$$

Under risk-neutral probability $\mathbb{Q}$, there exists a $\mathbb{Q}$-Brownian motion $B_t^{\mathbb{Q}}$ such that

interest rate

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}}).$$

$S_T = S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma B_T)$

$\Rightarrow S_T = S_t \exp((r - \frac{\sigma^2}{2})(T-t) + \sigma(B_T - B_t))$

Content Review

Question

For an option with payoff function $g(S_T)$, the option price is given by

interest rate

$$u(t, S_t) := \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(S_T)|S_t],$$

and $(e^{-rt}S_t)_{t\in[0,T]}$, $(e^{-rt}u(t,S_t))_{t\in[0,T]}$ are martingales.

$e^{-r(T-t)} f(B_T)$

Brownian motion

Question

$-ru$

If $u(t,x) := \mathbb{E}[f(B_T)|B_t = x]$, then $\partial_t u + \frac{1}{2}\partial^2_{xx}u = 0$.
If $u(t,x) := \mathbb{E}[f(X_T)|X_t = x]$, where $dX_t = \mu dt + \sigma dB_t$, then $\partial_t u + \frac{1}{2}\sigma^2\partial^2_{xx}u + \mu\partial_x u = 0$.
If $u(t,x) := \mathbb{E}[f(S_T)|S_t = x]$, where $dS_t = \mu S_t dt + \sigma S_t dB_t$, then $\partial_t u + \frac{1}{2}\sigma^2 x^2\partial^2_{xx}u + \mu x\partial_x u = 0$.

$-ru$

$e^{-r(T-t)} f(S_T) = v(t,x)$

Question

(Computation of intergral)

(1) $\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx$ $\quad N(0,1)\ \frac{1}{\sqrt{2\pi}}\int_R e^{-\frac{x^2}{2}}dx = 1 \Rightarrow \int_R e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}$

(2) $\int_{\mathbb{R}} \underbrace{x \cdot e^{-\frac{x^2}{2}}}_{\text{odd}}\, dx = 0$ $= \int_R e^{-\frac{x^2}{2}} d\left(\frac{x^2}{2}\right) = e^{-\frac{x^2}{2}}\cdot(-1)\Big|_{-\infty}^{\infty} = 0$

(3) $\int_{\mathbb{R}} \underbrace{x^2}_{x\cdot x} \cdot e^{-\frac{x^2}{2}}\, dx = \int_R x\cdot e^{-\frac{x^2}{2}} \cdot d\left(\frac{x^2}{2}\right) = \int_R x \cdot d\left(e^{-\frac{x^2}{2}}\right)\cdot(-1) = (-1)\cdot x\cdot e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty}$ $(=0)$ $+ \int_R e^{-\frac{x^2}{2}}\cdot dx = \sqrt{2\pi}$

(4) $\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{2b}}\, dx = \int_R e^{-\frac{y^2}{2}}\cdot\sqrt{b}\, dy = \sqrt{b}\cdot\sqrt{2\pi} = \sqrt{2\pi b}$ $\quad y = \frac{x-a}{\sqrt{b}}$

(5) $\int_{\mathbb{R}} e^x \cdot e^{-\frac{x^2}{2}}\, dx$

$= \int_R e^{-\frac{x^2-2x}{2}} dx \quad (x^2-2x = (x-1)^2 - 1)$

$= \int_R e^{-\frac{(x-1)^2-1}{2}} dx = e^{\frac{1}{2}}\cdot\int_R e^{-\frac{(x-1)^2}{2}} dx \overset{a=1,\ b=1}{=} e^{\frac{1}{2}}\cdot\sqrt{2\pi}$

Pricing by Martingale Approach

Question

Consider the stock price $(S_t)_{t\geq 0}$ which follows the Black Scholes model. Given risk-free interest rate r, find the price of a financial contract associated with S_t maturing at T with payoff function $g(x) := x^2$.

$g(S_T) = S_T^2$

Solution:

Since S_t follows the Black Scholes Model, we have

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t).$$

Under risk-neutral probability $\mathbb{Q}$, there exists a $\mathbb{Q}$-Brownian motion $B_t^{\mathbb{Q}}$ such that

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}}).$$

$\cancel{V \quad e^{-r(T-t)} \cdot V_T}$ $\qquad e^{-rt} V_t = E^{\mathbb{Q}}(e^{-rT} V_T \mid \mathcal{F}_t)$

Pricing by Martingale Approach

Denote by V_t the contract price at time $t \leq T$, the price of financial contract with payoff S_T^2. It is clear that $e^{-rt}V_t$ is a martingale under $\mathbb{Q}$. Notice that $V_T = g(S_T) = S_T^2$. Therefore,

$E(X_t | \mathcal{F}_s) = X_s, \; s \leq t$

$$e^{-rt}V_t = \mathbb{E}^{\mathbb{Q}}[e^{-rT}V_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T^2|\mathcal{F}_t].$$

$= S_t^2 \exp((r-\frac{\sigma^2}{2})(T-t)+\sigma(B_T-B_t)).$

$$\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}S_T^2|\mathcal{F}_t] \\
&= S_t^2 e^{-rT}\mathbb{E}^{\mathbb{Q}}[\exp((r-\sigma^2/2)(T-t)+\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))^2|\mathcal{F}_t] \\
&= \underbrace{S_t^2 e^{-rT+(2r-\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}[\exp(2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))|\mathcal{F}_t]}_{e^{2\sigma^2(T-t)}}.
\end{aligned}$$

Note that $B_T^{\mathbb{Q}} - B_t^{\mathbb{Q}}$ is independent of $\mathcal{F}_t$ with distribution $N(0, T-t)$ under $\mathbb{Q}$, then by characteristic function, we have

① characteristic function $X \sim N(\mu, \sigma^2) \Rightarrow E(e^X) = e^{\mu + \frac{\sigma^2}{2}}$

$$\mathbb{E}^{\mathbb{Q}}[\exp(2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[e^{2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}})}] = e^{2\sigma^2(T-t)}.$$

$Y \sim N(0, 4\sigma^2(T-t))$

Finally, $V_t = S_t^2 e^{(r+\sigma^2)(T-t)}$. Taking $t = 0$, we have $V_0 = S_0^2 e^{(r+\sigma^2)T}$.

$Y \quad E(e^{Y}) = e^{2\sigma^2(T-t)}$

$\Uparrow$

② $E\left(e^{2\sigma(B_T - B_t)}\right)$

$= E(e^{Y}) \qquad Y \sim N(0, 4\sigma^2(T-t))$

$$= \frac{1}{\sqrt{2\pi \cdot 4\sigma^2(T-t)}} \int_R e^{x} \cdot e^{-\frac{x^2}{2 \cdot 4\sigma^2(T-t)}} \, dx$$

$$= \frac{1}{\sqrt{8\pi\sigma^2(T-t)}} \int_R e^{-\frac{x^2 - 8\sigma^2(T-t)x}{8\sigma^2(T-t)}} \, dx$$

$$(x^2 - 8\sigma^2(T-t)x) = (x - 4\sigma^2(T-t))^2 - (4\sigma^2(T-t))^2$$

$$= \int_R e^{-\frac{(x-4\sigma^2(T-t))^2 - (4\sigma^2(T-t))^2}{8\sigma^2(T-t)}} \, dx$$

$$= e^{\frac{4\sigma^2(T-t)}{2}} \int_R e^{-\frac{(x-4\sigma^2(T-t))^2}{8\sigma^2(T-t)}} \, dx$$

$a = 4\sigma^2(T-t)$

$b = 4\sigma^2(T-t)$

$$= e^{2\sigma^2(T-t)} \cdot \sqrt{2\pi \cdot 4\sigma^2(T-t)}$$

$$= e^{2\sigma^2(T-t)}$$